Worksheet for Section 12.1

Section 12.1 is about *vectors* — a vector is a way of measuring quantities so that the measurement includes both *magnitude* (e.g. speed, such as 55 mph, or weight, such as 27 lbs.) and *direction* (e.g. north, up, left, etc.). Many physical measurements include direction in the measurements (e.g. velocity, force, acceleration), so they are represented by vectors. A vector can be represented graphically by a *directed line segment*, say from point P to point Q. Then the direction of the vector is represented by the direction from P to Q, and the magnitude by the length of the line segment, the distance from P to Q. However, vectors do *not* include information about *position* — any other line segment having the same length and direction (i.e. the line segments have the same slopes) represents the same vector. For example, graph the directed line segment from $P = (-1, 2)$ to $Q = (2, 0)$. (I have provided some grids below for you to do such graphing.) The point P is the *initial point* of the vector, and Q is its *terminal point*. Now graph the directed line segment from $R = (0, 0)$ to $S = (3, -2)$. Find the lengths of these two directed line segments, and the slopes of the lines. These two directed line segments represent the *same* vector, in spite of the fact that they are in different positions. The vector will be denoted $\vec{v}$. Writing a vector in its *component form*, $\vec{v} = \langle v_1, v_2\rangle$, eliminates the position information — for the vector from $P = (p_1, p_2)$ to $Q = (q_1, q_2)$, the component form of $\vec{v}$ is given by $\vec{v} = \langle v_1, v_2\rangle = \langle q_1 - p_1, q_2 - p_2\rangle$. Find the component form for the vector above from P to Q, and also from R to S. (To save space, the vector from P to Q is often denoted $\vec{PQ}$.) The *length* of a vector $\vec{v}$ is the length of the directed line segment: $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$. Find the lengths $\|\vec{PQ}\|$ and $\|\vec{RS}\|$ of the vectors $\vec{PQ}$ and $\vec{RS}$ above.

Vectors can be combined in several ways, leading to a whole vector arithmetic: for two vectors $\vec{u} = \langle u_1, u_2\rangle$ and $\vec{v} = \langle v1, v_2\rangle$,

- *Addition:* $\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2\rangle$ (sometimes called the *resultant vector* of $\vec{u}$ and $\vec{v}$)
- *Scalar Multiplication:* $c\vec{u} = \langle cu_1, cu_2\rangle$
- *Negative:* $-\vec{u} = \langle -u_1, -u_2\rangle$
- *Difference:* $\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2\rangle$

The properties of this vector arithmetic (p. 704) are very similar to the properties of ordinary arithmetic (commutative, distributive, etc.) but using these operations instead. The vector $\vec{0} = \langle 0, 0\rangle$ is called the *zero vector*, and plays a role in some of these properties. The properties of vector arithmetic can be proven, using the component form of vectors, from the corresponding properties of ordinary arithmetic. The vector operations have precise physical interpretations, too — for example, if two people are pulling on an object from two different directions, the net force being applied to the object is given by the resultant vector of the two separate force vectors being applied. As examples of using the operations, if $\vec{u} = \langle 2, 1\rangle$, $\vec{v} = \langle -2, 0\rangle$, and $\vec{w} = \langle 1, -1\rangle$, find the vector given by $2\vec{u} - \vec{v} + 3\vec{w}$.

For a vector $\vec{v}$, the length of the scalar multiple $c\vec{v}$ can be found from the length of $\vec{v}$: $\|c\vec{v}\| = |c|\|\vec{v}\|$. This leads to a way of finding a *unit vector* $\vec{u}$ in the direction of a given nonzero vector $\vec{v}$:

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\|\vec{v}\|}\vec{v}$$

This unit vector contains *only* direction information, not magnitude. For example, find the unit vector in the direction of $\vec{v} = \langle -2, 5\rangle$. The *standard unit vectors* $\vec{i} = \langle 1, 0\rangle$ (in the direction of the positive x-axis) and $\vec{j} = \langle 0, 1\rangle$ (in the direction of the positive y-axis) privide another way of representing vectors: the vector $\vec{v} = \langle v_1, v_2\rangle$ can be written as a *linear combination* of $\vec{i}$ and $\vec{j}$: $\vec{v} = v_1\vec{i} + v_2\vec{j}$. For this reason, the components v_1 and v_2 of $\vec{v}$ are often called the *horizontal* and *vertical* components, respectively. Furthermore, any unit vector can be drawn with its initial point at the origin and its terminal point on the unit circle, so it must be possible to represent such a unit vector $\vec{u}$ in the form $\vec{u} = \langle \cos\theta, \sin\theta\rangle$, where the angle θ is the angle that the vector $\vec{u}$ forms with the positive x-axis. This provides yet another way of writing vectors: $\vec{v} = \|\vec{v}\| \langle \cos\theta, \sin\theta\rangle = \|\vec{v}\| \cos\theta\vec{i} + \|\vec{v}\| \sin\theta\vec{j}$ For example, if the vector $\vec{v}$ has length 5 and forms an angle of $45° = \pi/4$ with the positive x-axis, then write $\vec{v}$ in the form $v_1\vec{i} + v_2\vec{j}$.

One last bit of vector-based geometry: by regarding two vectors $\vec{v}$ and $\vec{w}$ as two sides of a triangle, geometry of triangles leads to a fact which can be stated in vector terms — the Triangle Inequality:

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$$

(In triangle terminology, the sum of the lengths of two sides of a triangle is at least the length of the third side.) I will show you the vector picture of this in class.

I will work through a physical application of vectors in class as well.