Worksheet for Section 7.6

Section 7.6 is about improper integrals — that is, integrals that involve unbounded intervals (such as $[0, \infty)$) or unbounded integrands (such as the function $f(x) = 1/x$ on the interval $(0, 1]$).

Start with the first type — integrals with unbounded intervals of integration. There are two possibilities here: unbounded on the right, such as $[0, \infty)$, and unbounded on the left, such as $(-\infty, 0]$. These two possibilities are handled similarly, so look at the first type:

$$\int_a^\infty f(x)\, dx = \lim_{N\to\infty} \int_a^N f(x)\, dx$$

So the idea is to treat the improper integral as a limit of ordinary integrals. (*Note:* You do have to know that the integrals $\int_a^N f(x)\, dx$ exist for all $N \geq a$.) If the limit exists, the improper integral is said to *converge*; if not, it is said to *diverge*. Consider the following examples:

$$\int_1^\infty \frac{1}{x^2} dx \qquad \int_1^\infty \frac{1}{x} dx \qquad \int_1^\infty \frac{1}{x^p} dx$$

In the last example above, find conditions that specify for which values of p the integral will converge. The first two examples are actually cases of this last problem. Here are a few more examples:

$$\int_0^\infty xe^{-x}\, dx \qquad \int_{-\infty}^{-1} \frac{1}{x^3} dx \qquad \int_{-\infty}^{0} e^x\, dx$$

In the last two examples, the interval is unbounded on the left. How can you use a modification of the same procedure to find answers to these?

The second type of improper integral has to do with integrals of functions which are unbounded at some point (such as at a vertical asymptote of a rational function). The basic idea is similar: you want to evaluate the improper integral by treating it as a limit of ordinary integrals. Suppose $f(x)$ is unbounded at a (due to a vertical asymptote or some similar behavior) and you want to find $\int_a^b f(x)\, dx$. In spite of the unbounded behavior at a, it may still be possible to evaluate this improper integral, by treating it as a limit of ordinary integrals:

$$\int_a^b f(x)\, dx = \lim_{t\to a^+} \int_t^b f(x)\, dx$$

Again, it is important that the ordinary integrals $\int_t^b f(x)\, dx$ all exist — this is why the limit in this case is from *above* — t must approach a from *within* the interval.

Here are a few examples:

$$\int_0^2 \frac{1}{\sqrt{2-x}}dx \qquad \int_{\pi/2}^{\pi} \sec x \, dx \qquad \int_0^2 \frac{1}{x-1}dx$$

In the first problem, the unbounded point is at a *right* endpoint (the upper end of the interval) — using the same idea, how would you set up this limit? Be extra careful in the last problem — the unbounded point is in the *interior* of the interval.