Worksheet for Section 12.4

Section 12.4 is about another way of multiplying vectors, called a *cross product*. The cross product of two vectors $\vec{u}$ and $\vec{v}$ makes sense only if both vectors are vectors in space — there is no similar product for vectors in the plane. For vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, the cross product $\vec{u} \times \vec{v}$ is defined as follows:

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

The book also mentions a way of computing a cross product (and remembering the order of the terms) using a determinant of a matrix — since I don’t want to assume you have seen matrix determinants before, though, I will not be using that form. The algebraic properties of the cross product are in many ways similar to those of the dot product, but there are some very important differences, as well. These algebraic properties are somewhat harder to prove for the cross product because of the nature of the formula. Perhaps more revealing, though, are the *geometric* properties of the cross product:

- $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin \theta$, where $\theta$ is the angle between $\vec{u}$ and $\vec{v}$
- $\vec{u} \times \vec{v} = \vec{0}$ if and only if $\vec{u}$ and $\vec{v}$ are parallel
- $\|\vec{u} \times \vec{v}\|$ is the area of the parallelogram having $\vec{u}$ and $\vec{v}$ as adjacent sides

For example, compute the cross product of $\vec{u} = \langle 1, 2, 0 \rangle$ and $\vec{v} = \langle 2, -3, 1 \rangle$. Can you use this cross product to find a *unit* vector which is orthogonal to both $\vec{u}$ and $\vec{v}$? As a geometric application of the cross product, verify that the four points $A = (1, 1, 2)$, $B = (2, 0, 3)$, $C = (3, 3, -1)$, and $D = (2, 4, -2)$ are the vertices of a parallelogram, and compute the area of that parallelogram. Is the parallelogram a rectangle?

Another kind of product of vectors, called a *triple scalar product*, involves three vectors in space, and can be computed using the dot product and cross product: the triple scalar product of $\vec{u}$, $\vec{v}$, and $\vec{w}$ is given by $\vec{u} \cdot (\vec{v} \times \vec{w})$. This triple scalar product again has an interesting geometric interpretation: $|\vec{u} \cdot (\vec{v} \times \vec{w})|$ gives the *volume* of the parallelepiped with the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ as adjacent sides (a parallelepiped is a sort of oblique box — a parallelepiped is to a cube in space what a parallelogram is to a square in the plane). Find the volume of the parallelepiped having the vectors $\vec{u} = \langle 1, 2, 0 \rangle$, $\vec{v} = \langle 2, -3, 1 \rangle$, and $\vec{w} = \langle 1, 1, -1 \rangle$ as adjacent sides. Also, compute the areas of the faces of the parallelepiped, using cross products. (The parallelepiped has six faces, but opposite faces have equal area, so you only need to compute three areas.)

One useful application of the cross product in physics has to do with computing *torque* — I will look at an example in class.