Worksheet for Section 6.2

Section 6.2 is about various applications of integrals, in particular applications which involve new interpretations of Riemann sums. In many cases, there are interesting pictures which go with these interpretations — I will show you the pictures in class.

A solid with cross-sectional area $A(x)$ for cross-sections perpendicular to the x-axis on the interval $[a, b]$ has volume given by the following formula:

$$V = \int_a^b A(x)\,dx$$

For example, horizontal cross-sections of a regular square-based pyramid are squares — it is easy to find the area of a square, given the length of a side of the square. Suppose you want to find the volume of a pyramid with base side length L and height H. Start by finding a formula for the area of a cross-section square that is height h above the base (you will need to find the length of a side of the cross-section square, using similar triangle geometry). Then find the volume of the pyramid by integrating this area formula with respect to h, for h in the interval $[0, H]$.

If the function $\rho(x)$ gives the linear mass density of a wire or rod (usually given in units like kilograms per meter), where the rod lies along the x-axis from $x = a$ to $x = b$, then the total mass M of the rod is given by:

$$M = \int_a^b \rho(x)\,dx$$

For example, suppose a 5-meter rod has linear density given by $\rho(x) = 2 + 5x - x^2$ (kg/m) for x in the interval $[0, 5]$. Find the total mass of the rod.

In some cases, the density depends on the distance r from the origin. This sort of *radial* density function is particularly useful for modeling population of a city, where the population density is a function of the distance from the city center. (Population density is measured in units of thousands of people per square mile, or something similar.) If $\rho(r)$ is the radial population density function, then the total population P within radius R of the city center is given by:

$$P = 2\pi \int_0^R r \cdot \rho(r)\,dr$$

For example, suppose $\rho(r) = \dfrac{15}{\sqrt{1 + r^2}}$ (where r is measured in miles). Find the total population within a twenty mile radius of the city center.

If flow of liquid through a circular tube or pipe is *laminar* (i.e. not turbulent), then the velocity of the fluid flow depends only on the distance r from the center of the tube — friction between the fluid and the side of the tube would tend to slow the flow at that boundary, and the viscosity of the liquid would make that slowing affect the flow further inside the tube. If the flow is described by the velocity function $v(r)$ then the total flow rate Q (measured e.g. in milliliters per second) is given by:

$$Q = 2\pi \int_0^R r \cdot v(r)\,dr$$

For example the velocity of blood flowing through a blood vessel of radius R is given by Poiseuille's Law, $v(r) = k(R^2 - r^2)$, where r is the distance from the center of the vessel (in centimeters) and k is constant (accounting for such things as the viscosity of the blood). If $k = 0.5$, find the total flow rate Q of blood through the blood vessel. (Your answer will depend on R.)

The *average value* of a function $f(x)$ on an interval $[a, b]$ is given by:

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

For example, find the average value of $f(x) = 2x^2 + 1$ on the interval $[-1, 3]$. Graph $f(x)$ on $[-1, 3]$, and draw a horizontal line corresponding to your computed average value — what does the graph indicate about the average value of $f(x)$? Now find the average value of $\sin x$ on $[0, \pi]$, and again graph the function and a horizontal line at the average value.

As the name suggests, the Mean Value Theorem for Integrals is closely related to the Mean Value Theorem from last semester — but that Mean Value Theorem had to do with derivatives. Recall that version of the Mean Value Theorem: if $f(x)$ is continuous on the interval $[a, b]$ and differentiable on (a, b), then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The right hand side of this equation represents the *average rate of change* of f on $[a, b]$ — hence "mean" value.

The Mean Value Theorem for Integrals says something very similar: if $f(x)$ is continuous on $[a, b]$, then there is a number c in (a, b) such that

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

(There is an interesting picture that goes along with this Theorem — I will show it to you in class.) Again, as with the original Mean Value Theorem, this says nothing about how to find c — it just says that there has to be such a number.

In some cases, you can find the value of c as guaranteed by the Theorem. Suppose, for example, that you want to find the value of c for the following definite integral:

$$\int_0^9 \sqrt{x}\,dx$$

Start by computing the definite integral, then plug it, along with the limits of the integral, into the formula from the Theorem, and solve for c. Why does the Theorem not apply to the function $f(x) = \frac{1}{x}$ on the interval $(0, 2)$?