Worksheet for Section 10.7

Section 10.7 is about Taylor series and Maclaurin series — the series that come from treating the Taylor and Maclaurin polynomials of Section 8.4 as partial sums. It turns out that if a function \( f(x) \) is a function which can be represented by a convergent power series, so that \( f(x) = \sum a_n(x - c)^n \), then the coefficients \( a_n \) must have a special form: \( a_n = f^{(n)}(c)/n! \) — exactly the same as the coefficients of the Taylor polynomials! So the power series obtained for a function \( f(x) \) by using these coefficients is called a Taylor series:

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots
\]

If \( c = 0 \), you get at Maclaurin series:

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots
\]

Using these formulas, find Maclaurin series for \( f(x) = e^x \) and \( g(x) = \sin x \), and a Taylor series centered at 1 for \( h(x) = \ln x \).

You can determine convergence of these Taylor series by the same techniques you have used before (start with the Ratio Test), but there is another way for Taylor series. A Taylor series for \( f(x) \) centered at \( c \) converges to \( f(x) \) (for every \( x \) in an interval \( I \)) just in case the limit of the Taylor remainder is zero:

\[
\lim_{n \to \infty} R_n(x) = \lim_{n \to \infty} \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}
\]

for some \( z \) between \( x \) and \( c \) (and for every \( x \) in \( I \)). Using this, show that the Maclaurin series you found above for \( g(x) = \sin x \) converges for all \( x \). How can you use this series, together with the Operations on Series from Section 8.9, to find a power series for \( h(x) = \sin(x^2) \)?

There is a table on p. 628 in the book of power series for several elementary functions. Using these series, find power series representations for each of the following functions:

\[
\begin{align*}
 f(x) &= \sqrt{1 + x}  & g(x) &= \sqrt{1 - x^2}  & h(x) &= \cos^2 x  & j(x) &= e^{-x^2}
\end{align*}
\]

The first two examples above use the binomial series for the function \((1 + x)^k\) in the table — this series doesn’t look useful until you notice, as in these examples, that \( k \) doesn’t have to be a positive integer (in the examples, \( k = \frac{1}{2} \)).

Using the last example above, find an estimate for the following definite integral:

\[
\int_0^1 e^{-x^2} \, dx
\]

How can you determine the accuracy of your estimate?