Worksheet for Section 10.4

Section 10.4 is about series which may have negative terms. If \( \sum b_n \) is such a series, perhaps tests such as the Limit Comparison Test can be applied to the series \( \sum |b_n| \) to determine if it converges. But can this be used to determine if the original series \( \sum b_n \) converges? In fact, you can say quite a bit:

- If \( \sum |b_n| \) converges, then \( \sum b_n \) converges.

The reverse is not necessarily true, though — for example, consider the series

\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}
\]

In this case, the series \( \sum b_n \) converges, but \( \sum |b_n| \) does not. This makes possible a distinction summarized in the following definitions:

- \( \sum b_n \) is absolutely convergent if \( \sum |b_n| \) converges
- \( \sum b_n \) is conditionally convergent if \( \sum b_n \) converges but \( \sum |b_n| \) does not.

So the series \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \) is conditionally convergent. For each of the following series, classify the series as absolutely convergent, conditionally convergent, or divergent:

\[
\begin{align*}
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} & \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{3n+5} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \\
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} & \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}
\end{align*}
\]

In an alternating series, consecutive terms have different signs, so the signs alternate. It is important that this alternation happen with every term, so that the series must have one of the following forms:

\[
\sum_{n=1}^{\infty} (-1)^n a_n \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n
\]

where \( a_n > 0 \) for all \( n \). In this situation, the test for convergence of the series is both special and simple — the series converges if and only if both of the following conditions are met:

- \( a_{n+1} \leq a_n \) for all \( n \) (sequence \( \{a_n\} \) is monotonic, decreasing)
- \( \lim_{n \to \infty} a_n = 0 \) (limit of sequence \( \{a_n\} \) is 0)

(The second condition above is much like the \( n^{th} \) Term Test, but now you can conclude that the series converges.) Using this test, determine which of the following alternating series converge or diverge:

\[
\begin{align*}
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} & \quad \sum_{n=1}^{\infty} \frac{n^2}{(-2)^n} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^2 - 3n + 1} \\
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} & \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n
\end{align*}
\]

Note that, for the alternating series \( \sum (-1)^n a_n \), if \( \sum a_n \) converges, then the alternating series converges absolutely, in the sense defined above. Absolute convergence...
can be determined using the tests from the previous section. Which of the above alternating series converge absolutely?

There is also a way to approximate the sum of an alternating series. Suppose the alternating series \( \sum_{n=1}^{\infty} (-1)^n a_n \) converges, so that \( \sum_{n=1}^{\infty} (-1)^n a_n = S \). Let \( S_N \) be the \( N^{th} \) partial sum of the series. Then the remainder \( R_N \) is given by \( R_N = S - S_N \) — this measures the difference between the partial sum and the actual sum. This remainder can be approximated by the following inequality:

\[
|S - S_N| = |R_N| \leq a_{N+1}
\]

Using this inequality, approximate the sum for each of the following series by finding \( S_6 \) and determining how accurate \( S_6 \) is as an approximation to the actual sum:

\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}
\]