Worksheet for Section 10.3

Section 10.3 is about the several tests for convergence of series — the Integral Test and two Comparison Tests. These tests apply to series with positive terms, so that the partial sums form an increasing sequence. Using the facts about monotonic and bounded sequences from Section 10.1 gives the following “dichotomy” for positive series:

- if the partial sums are bounded above, then the series converges;
- if the partial sums are not bounded above, then the series diverges.

Note that this only works for positive series. We will deal with series with negative terms in the next section.

The Integral Test establishes a relationship between infinite series and improper integrals (Section 7.7). Consider first an example — the harmonic series:

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]

I stated, without showing, in the last section that this series diverges. There are two ways to show this. The first method, a direct approach involving a direct comparison of terms of the series, I will show you in class. This approach does not generalize well to other series, though. For example, can you find a way to determine if the following series converges?

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots \]

The approach for the harmonic series does not work — in fact, this series converges. Why is there such a sharp difference between these similar series?

The second approach, however, does provide a general way of looking at series such as those above: The Integral Test. If a series \( \sum_{n=1}^{\infty} a_n \) is given by \( a_n = f(n) \) for all positive integers \( n \), and the function \( f(x) \) is positive, decreasing, and continuous for \( x \geq 1 \), then:

\[ \sum_{n=1}^{\infty} a_n \text{ converges if and only if } \int_{1}^{\infty} f(x) \, dx \text{ converges} \]

(There is an interesting picture that accompanies this Test, and I will also show that to you in class.) One thing to be careful of when using the Integral Test: the Test can tell you whether or not a series converges, but if it does converge, the Test does not give you the value of the sum. Use the Integral Test to determine which of the following series converge:

\[ \sum_{n=1}^{\infty} \frac{1}{n} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \sum_{n=1}^{\infty} \frac{n}{3n^2 + 2} \quad \sum_{n=1}^{\infty} \frac{1}{n^2 + 4} \]

Using the Integral Test, can you determine for which values of \( p \) the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges?

Series such as this are called \( p \)-series.

There are two techniques for comparing series, in particular to use information about one series (whether it converges or diverges) to determine the same information about another series.
The first way to compare two series is to directly compare corresponding terms of the two series. The Direct Comparison Test summarizes what you can conclude from such a comparison: for two series \( \sum a_n \) and \( \sum b_n \), suppose \( 0 < a_n \leq b_n \) for all \( n \) — in this case, if \( \sum b_n \) converges, then so does \( \sum a_n \); if \( \sum a_n \) diverges, so does \( \sum b_n \). (Notice what is happening here: if the sequence with larger terms converges, so does the sequence with smaller terms; on the other hand, if the sequence with smaller terms diverges, so does the sequence with larger terms.) When using the Direct Comparison Test, you usually want to compare a series you don’t know about to one you do know about, but which resembles the first series. For example, determine whether the series \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 3} \) converges or diverges, by direct comparison with the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) (which is a \( p \)-series). Use similar techniques in each of the following examples:

\[
\sum_{n=1}^{\infty} \frac{1}{2^n + n} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} - 1} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + 1} \quad \sum_{n=1}^{\infty} \frac{1}{n^2 - 1}
\]

In the last two examples above, you have to be extra careful about how to choose the series for comparison. There is another type of comparison test that makes situations like this easier: The Limit Comparison Test. For two series \( \sum a_n \) and \( \sum b_n \) with \( a_n > 0 \) and \( b_n > 0 \), suppose

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = L
\]

where \( L \) is finite and positive, then either both series converge or both of them diverge. If \( L = 0 \) or \( L = \infty \), then the test is inconclusive. Use the Limit Comparison Test to compare the series \( \sum_{n=1}^{\infty} \frac{1}{n^2 - 1} \) with the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \). This example shows one of the important guidelines for using the Limit Comparison Test: to compare a series with a \( p \)-series, choose the \( p \)-series whose degree matches the total degree of the original series. This is a good way to determine convergence of “messy” series involving rational expressions, by comparing them with “neat” \( p \)-series. Here are some more examples to work out using the Limit Comparison Test:

\[
\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + 1} \quad \sum_{n=1}^{\infty} \frac{1}{n^3 - 2n^2 + 3} \quad \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3 - 2n^2 + 3} \quad \sum_{n=1}^{\infty} \frac{2^n}{2n^2 + 4}
\]