Worksheet for Section 10.2

Having looked at sequences in Section 10.1, it is now time to look at series in this section. A series is (by definition) the sum of the terms of a sequence. The notation for such a sum is similar to what you used for the sum of terms in a Riemann Sum:

\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots \]

So the terms of the sequence \( \{a_n\} \) are being added together. In spite of the fact that this sum has infinitely many terms, it is entirely possible that such a series may add up to a finite number. Consider, for example, the following series:

\[ \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \]

To determine if this series has a sum, consider the sequence of partial sums: \( S_1 = \frac{1}{2}, S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}, \ldots, S_n = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} \). This sequence \( \{S_n\} \) has a limit: \( \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{2^n - 1}{2^n} = 1 \). Since this is the case, the infinite series \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) is said to converge to this limit: \( \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1 \).

This procedure of considering the sequence of partial sums is important, and can be used to define the sum of an infinite series. For a series \( \sum_{n=1}^{\infty} a_n \), define the sequence of partial sums: \( S_1 = a_1, S_2 = a_1 + a_2, S_3 = a_1 + a_2 + a_3, \) etc., \( S_n = a_1 + a_2 + \cdots + a_n \), for the sequence \( \{S_n\} \). If \( \{S_n\} \) converges to \( S \) (so that \( \lim_{n \to \infty} S_n = S \)), then the series \( \sum_{n=1}^{\infty} a_n \) is said to converge, and \( \sum_{n=1}^{\infty} a_n = S \). If \( \{S_n\} \) diverges, then the series diverges, too. Using the limit of the partial sums, determine if the following series converge, and if they do, find their limits:

\[ \sum_{n=1}^{\infty} \frac{1}{3^n}, \quad \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right), \quad \sum_{n=1}^{\infty} 2, \quad \sum_{n=1}^{\infty} n \]

The series \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) and \( \sum_{n=1}^{\infty} \frac{1}{3^n} \) are examples of geometric series; the series \( \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) \) is an example of a telescoping series.

The telescoping series don’t appear very often, but they are interesting examples for the algebra involved in finding their limits. For example, the series \( \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \)
is a telescoping series — to see this, take the expression \( \frac{1}{n(n+2)} \) and rewrite it using the technique of Partial Fractions, then write out the first few partial sums and see what happens to them.

Geometric series are a much more common example — in fact, they come up in many different physical situations, from straight line motion (the way to resolve Zeno’s Paradox about the path of an arrow is to consider the sum of a geometric series) to the path of a bouncing ball. A geometric series is a series of the following form:

\[
\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots
\]

(The fact that \( n \) starts at 0 instead of 1 doesn’t really matter — it is still an infinite series, and you can still write down a sequence of partial sums, the series just has an extra term at the beginning.) To see how this series works out, consider the partial sums:

\[
S_n = a + ar + ar^2 + \cdots + ar^n = a(1 + r + r^2 + \cdots + r^n) = a \frac{1 - r^n}{1 - r}
\]

If \( 0 < |r| < 1 \), \( r^n \to 0 \) as \( n \to \infty \), so \( \lim_{n \to \infty} S_n = \frac{a}{1 - r} = \sum_{n=0}^{\infty} ar^n \).

This gives a very useful formula for the sum of a geometric series. How would you use this formula to find the sums of the two geometric series above? What happens if \( |r| > 1 \), though? Consider the series \( \sum_{n=0}^{\infty} \left( \frac{3}{2} \right)^n \) — does this series have a limit? Geometric series can also help in writing repeating decimals as fractions — for example, how would you write the decimal \( 0.17171717 \ldots \) as a fraction?

The Properties of Series on p. 562 can help you figure out what is happening with certain combinations of series. Notice, however, that there is no property for the term-by-term product of two series — products just don’t work out that way. More important than these properties, though, is the \( n^{th} \)-Term Test for divergence of a series. First notice that if \( \sum_{n=0}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \). Therefore, if \( \lim_{n \to \infty} a_n \neq 0 \), then the series cannot converge. Two of the examples above work this way: \( \sum_{n=1}^{\infty} 2 \) and \( \sum_{n=1}^{\infty} n \) both diverge, because the limits of the terms are not 0. However, for the series \( \sum_{n=1}^{\infty} \frac{1}{n} \), the limit of the terms is 0, but (as we will see in the next section) this series diverges — this test does not work in reverse.